

# THE CRAMER-WOLD THEOREM ON QUADRATIC SURFACES AND HEISENBERG UNIQUENESS PAIRS

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**ABSTRACT.** Two measurable sets  $S, \Lambda \subseteq \mathbb{R}^d$  form a Heisenberg uniqueness pair, if every bounded measure  $\mu$  with support in  $S$  whose Fourier transform vanishes on  $\Lambda$  must be zero. We show that a quadratic hypersurface and the union of two hyperplanes in general position form a Heisenberg uniqueness pair in  $\mathbb{R}^d$ . As a corollary we obtain a new, surprising version of the classical Cramér-Wold theorem: a bounded measure supported on a quadratic hypersurface is uniquely determined by its projections onto two generic hyperplanes (whereas an arbitrary measure requires the knowledge of a dense set of projections). We also give an application to the unique continuation of eigenfunctions of second-order PDEs with constant coefficients.

## 1. INTRODUCTION

The notion of a Heisenberg uniqueness pair by Hedenmalm and Montes-Rodríguez [HMR] introduced a new facet to the investigation of Fourier transform pairs  $(\mu, \widehat{\mu})$  for a bounded Borel measure  $\mu$  and its Fourier transform (or characteristic function)

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} d\mu(x), \quad \xi \in \mathbb{R}^d.$$

In general a version of the uncertainty principle balances the size (of the support or of some quantity) of  $\mu$  with the size of  $\widehat{\mu}$ , so that both cannot be too small. By contrast, Heisenberg uniqueness pairs balance the size of the support of a measure  $\mu$  with the size of the zero set of  $\widehat{\mu}$ . A particular aspect is that both the support of  $\mu$  and the zero set of  $\widehat{\mu}$  may be singular sets.

### 1.1. Heisenberg Uniqueness Pairs.

To be precise, let  $S, \Lambda \subseteq \mathbb{R}^d$  be two measurable sets and let  $\mathcal{M}(S)$  denote the set of *finite* signed Borel measures supported in  $S$ . The following definition was first introduced in [HMR] in a slightly more restrictive form.

**Definition 1.** The pair  $(S, \Lambda) \subset \mathbb{R}^d \times \mathbb{R}^d$  is said to be a *Heisenberg uniqueness pair*, if the only measure  $\mu \in \mathcal{M}(S)$  such that  $\widehat{\mu} = 0$  on  $\Lambda$  is the measure  $\mu = 0$ .

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For a subset  $\mathcal{C} \subset \mathcal{M}(S)$  of measures,  $(S, \Lambda)$  is said to be a  $\mathcal{C}$ -Heisenberg uniqueness pair if the only measure  $\mu \in \mathcal{C}$  such that  $\widehat{\mu} = 0$  on  $\Lambda$  is the measure  $\mu = 0$ .

Equivalently, if two finite measures supported on  $S$  have characteristic functions agreeing on  $\Lambda$ , then they are equal.

If  $S$  is a smooth manifold, we write  $\mathcal{AC}(S)$  for the set of finite measures that are absolutely continuous with respect to the surface measure on  $S$ . The original concept in [HMR] was that of an  $\mathcal{AC}(S)$ -Heisenberg uniqueness pair and so far was almost exclusively studied in dimension  $d = 2$  when  $S$  is a smooth curve.

Since a general characterization of Heisenberg uniqueness pairs is out of reach, research so far has focussed on the investigation of specific examples, for instance  $S$  being a hyperbola [HMR], a circle [Le, Sj1], a parabola [Sj2], and  $\Lambda$  a set of lines or a discrete subset thereof, some cases with three parallel lines are treated in [Ba], the case of both  $S$  and  $\Lambda$  being circles or spheres is treated in [Le, Sj1, Sri].

In [JK] most of these examples were unified and extended by means of a new technique. The question of whether  $(S, \ell_1 \cup \ell_2)$  is a  $\mathcal{AC}(S)$ -Heisenberg uniqueness pair can then be answered by studying a certain dynamical system on  $S$  defined by two (distinct) lines  $\Lambda = \ell_1 \cup \ell_2$ .<sup>\*</sup> With this technique, one can derive a complete characterization in the case of conic sections  $S = \{(x, y) \in \mathbb{R}^2 : ax^2 + bxy + cy^2 + dx + ey = f\} \subseteq \mathbb{R}^2$  with  $a, b, c, d, e, f \in \mathbb{R}$  not all 0 and two lines  $\Lambda = \ell_1 \cup \ell_2$ . Before a description of these results, we make a convenient reduction. As already observed in [HMR], the notion of Heisenberg uniqueness pairs is invariant with respect to affine linear transformations, namely:

- [Inv 1] Fix  $x, \xi \in \mathbb{R}^d$ . Then  $(S, \Lambda)$  is a Heisenberg uniqueness pair, if and only if  $(S - x, \Lambda - \xi)$  is a Heisenberg uniqueness pair.
- [Inv 2] Fix  $T$  a linear invertible transformation  $\mathbb{R}^d \rightarrow \mathbb{R}^d$  and denote by  $T^*$  its adjoint. Then  $(S, \Lambda)$  is a Heisenberg uniqueness pair, if and only if  $(T^{-1}(S), T^*(\Lambda))$  is a Heisenberg uniqueness pair.

According to the invariance properties [Inv1-Inv2] it is enough to consider the following cases for the analysis of conic sections in  $\mathbb{R}^2$ .

- (i) **A single line.** If  $\Gamma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$  and  $\ell_1, \ell_2$  are two arbitrary lines through  $(0, 0)$ , then  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair. Actually  $(\Gamma, \ell_1)$  is already a Heisenberg uniqueness pair unless  $\ell_1$  is the line orthogonal to  $\Gamma$  [HMR].
- (ii) **The parabola.** If  $\Gamma = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$  and  $\ell_1, \ell_2$  are two arbitrary lines through  $(0, 0)$ , then  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair [Le, Sj2].

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<sup>\*</sup>Throughout the paper, two lines will always mean two *distinct* lines.

- (iii) **The circle.** If  $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  and  $\ell_1, \ell_2$  are two arbitrary lines through  $(0, 0)$  which intersect with an angle  $\theta \notin \pi\mathbb{Q}$ , then  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair [Le, Sj1].<sup>†</sup>
- (iv) **The hyperbola.** If  $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$  and  $\ell_1, \ell_2$  are two lines through  $(0, 0)$ . If  $\ell_1 = \mathbb{R}(a, b)$  and  $\ell_2 \neq \mathbb{R}(-a, b)$ , then  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair [HMR].

Further degenerate quadratic curves which were not discussed in [JK], but for which the techniques of [JK] apply are the following.

- (v) **The cone**  $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ . If  $\ell_1 = \mathbb{R}(a, b)$  and  $\ell_2 \neq \mathbb{R}(-a, b)$ , then  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair.
- (vi) **Two parallel lines**  $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 = a^2\}$ . If  $\ell_1, \ell_2$  are two arbitrary lines through  $(0, 0)$  except the  $y$  axis,  $(\Gamma, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair.

By applying the invariance properties of Heisenberg uniqueness pairs, we may summarize these results as follows.

**Theorem 1.1.** *Let  $Q$  be a quadratic form on  $\mathbb{R}^2$ ,  $v \in \mathbb{R}^2$ ,  $\rho \in \mathbb{R}$ , and  $S = \{(x, y) \in \mathbb{R}^2 : Q(x, y) + 2\langle v, (x, y) \rangle = \rho\}$ . Then there exists an exceptional set  $\mathcal{E} = \mathcal{E}(Q, v, \rho)$  of pairs of distinct directions, such that when  $\ell_1, \ell_2$  are two lines with directions not in  $\mathcal{E}$ , then  $(S, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair. Moreover, if  $\ell_1$  is fixed, then the set  $\{\ell_2 : (\ell_1, \ell_2) \notin \mathcal{E}\}$  is at most countable. Therefore  $\mathcal{E}$  has measure zero with respect to the surface measure on  $\mathbb{S}^1 \times \mathbb{S}^1$ .*

**Remark 1.** The dependence of the exceptional set on the parameters  $Q, v, \rho$  is simple. For  $S$  a parabola, a point (= degenerate ellipse), or two parallel lines (= degenerate parabola),  $\mathcal{E}$  is the empty set. In all other cases  $\mathcal{E}(Q, v, \rho)$  is a fixed set of directions that depends only on  $Q$ , but not on  $v$  and  $\rho$ . Thus for a fixed quadratic form  $Q$  only a set  $\mathcal{E}(Q)$  and  $\emptyset$  may occur.

In this paper we present the first complete study of Heisenberg uniqueness pairs in higher dimensions and will extend Theorem 1.1 to higher dimensions for the case of a quadratic hypersurface  $S$  and a union of hyperplanes  $\Lambda$ . We then make the connection with the classical Cramér-Wold Theorem which characterizes probability measures by their projections onto hyperplanes. In contrast to general measures, a measure supported on a quadratic hypersurface is uniquely determined already by its projection to two generic hyperplanes. Finally, as in [HMR] we interpret the notion of Heisenberg uniqueness pair on a quadratic hypersurface as a statement about the solutions to a partial differential equation of order two with constant coefficients.

Let us now describe our results with more precision. To start, let  $S$  be a quadratic hypersurface, that is a surface of the form  $S = \{x \in \mathbb{R}^d : P(x) = 0\}$  where  $P$  is a polynomial of total degree 2. In the following we always write  $P$  as the sum of a quadratic form  $Q$  and an affine form. This means that there exists a bilinear form  $B : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ , such that  $Q(x) = B(x, x)$  and

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<sup>†</sup>Actually, in [JK], it is shown that  $(\Gamma, \ell_1 \cup \ell_2)$  is an  $\mathcal{AC}(\Gamma)$ -Heisenberg uniqueness pairs. We will give a slightly simpler proof below that allows to extend the result.

a vector  $v \in \mathbb{R}^d$  and  $\rho \in \mathbb{R}$ , such that  $P(x) = Q(x) + 2\langle v, x \rangle - \rho$ . The set  $\Lambda$  will consist of two distinct hyperplanes  $H_1, H_2$ . We describe a hyperplane by a normal vector  $u \in \mathbb{R}^d$  with unit norm  $|u| = 1$  and the offset parameter  $s \in \mathbb{R}$ , and define

$$H_{u,s} = \{x \in \mathbb{R}^d : \langle x, u \rangle = s\} \quad \text{and} \quad H_u = H_{u,0}.$$

We first consider the case of two intersecting hyperplanes. Our main result is the following.

**Theorem 1.2.** *Let  $Q$  be a quadratic form on  $\mathbb{R}^d$ ,  $v \in \mathbb{R}^d$ ,  $\rho \in \mathbb{R}$  and  $S = \{x \in \mathbb{R}^d : Q(x) + 2\langle v, x \rangle = \rho\}$ . There exists an exceptional set  $\mathcal{E} = \mathcal{E}(Q, v, \rho)$  of pairs of distinct directions such that*

- (i) *the set  $\mathcal{E}$  has measure zero with respect to the surface measure on  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ ;*
- (ii) *when  $u_1, u_2 \in \mathbb{R}^d$  satisfy  $Q(u_1), Q(u_2) \neq 0$  and  $(u_1, u_2) \notin \mathcal{E}$ , then  $(S, H_{u_1} \cup H_{u_2})$  is a Heisenberg uniqueness pair.*

This result is very general and the proof gives an explicit construction of the exceptional set in terms of the exceptional sets described in the special cases of Theorem 1.1. Also notice that Theorem 1.2 holds for arbitrary measures, whereas the cited results in dimension  $d = 2$  were proved only for absolutely continuous measures. The proof of Theorem 1.2 will be given in Section 2. We will significantly extend the approach with dynamical systems [JK].

We will then focus on some specific cases and derive more precise results. We will also consider the case of parallel hyperplanes. In Section 3 we study the case when  $S$  is a cone or a hyperboloid in  $\mathbb{R}^d$  and  $\Lambda$  consists of two or more hyperplanes. In Section 4 we study the case when  $S$  is a sphere in  $\mathbb{R}^d$  and  $\Lambda$  consists either of two parallel hyperplanes or several hyperplanes with a common intersection. It is a curious feature of the dynamical systems approach that infinite Coxeter groups will occur naturally. Indeed, we will apply a deep theorem about such groups in the proof of Theorem 4.2.

Let us now see how our results can be interpreted in probability theory and PDEs.

## 1.2. A Cramér-Wold type theorem.

Let  $X, Y$  be two metric spaces and  $f : X \rightarrow Y$  a Borel mapping. Recall that, if  $\mu$  is a measure on  $X$ , the push-forward of  $\mu$  by  $f$  is the measure  $f_*\mu$  on  $Y$  defined by  $f_*\mu(E) = \mu(f^{-1}(E))$  and that, for every continuous compactly supported function  $g$  on  $Y$ ,

$$(1.1) \quad \int_Y g(y) \, df_*\mu(y) = \int_X g(f(x)) \, d\mu(x).$$

The classical Cramér-Wold Theorem [CW] asserts that a probability measure on  $\mathbb{R}^d$  is uniquely determined by the set

$$\{\pi_*\mu : \pi \text{ a projection on a hyperplane through } 0\},$$

that is, if  $\pi_*\mu = \pi_*\nu$  for every  $\pi$ , or equivalently  $\pi_*(\mu - \nu) = 0$ , then  $\mu - \nu = 0$ . This fact is easily proven as follows: if  $\pi$  is the projection on a hyperplane  $H$ ,

then the Fourier transform of  $\pi_*(\mu - \nu)$  in  $H$  is just the restriction of  $\widehat{\mu - \nu}$  to  $H$  (see (ii) in Lemma 2.2). Therefore,  $\pi_*(\mu - \nu) = 0$  for every  $H$  if and only if  $\widehat{\mu - \nu} = 0$  which occurs if and only if  $\mu - \nu = 0$ . As  $\widehat{\mu - \nu}$  is continuous one only needs a dense set of hyperplanes.

In general, not much more can be said, see e.g. [BMR, Gi, He, Re]. In particular, finitely many projections do never determine a measure on  $\mathbb{R}^d$  completely. However, if we restrict the support of a measure to a quadratic hypersurface, we obtain a completely different version of a Cramér-Wold theorem. The following statement is an immediate consequence of Theorem 1.2.

**Theorem 1.3.** *Let  $Q$  be a quadratic form on  $\mathbb{R}^d$ ,  $v \in \mathbb{R}^d$ ,  $\rho \in \mathbb{R}$  and  $S = \{x \in \mathbb{R}^d : Q(x) + 2\langle v, x \rangle = \rho\}$ . Then there exists an exceptional set  $\mathcal{E} = \mathcal{E}(Q, v, \rho)$  of measure zero in  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  with the following property: Let  $u_1, u_2 \in \mathbb{R}^d$  be distinct directions, such that  $Q(u_1), Q(u_2) \neq 0$  and  $(u_1, u_2) \notin \mathcal{E}$ . If  $\mu, \nu \in \mathcal{M}(S)$  and  $\pi_{H_{u_1}} * \mu = \pi_{H_{u_1}} * \nu$  and  $\pi_{H_{u_2}} * \mu = \pi_{H_{u_2}} * \nu$ , then  $\mu = \nu$ .*

Thus a finite measure supported on a quadratic hypersurface is determined uniquely by its projections to two generic hyperplanes.

### 1.3. Applications to linear PDEs.

Let  $P$  be a quadratic polynomial on  $\mathbb{R}^d$  and  $S = \{x \in \mathbb{R}^d : P(x) = 0\}$ . Replacing each variable  $x_j$  by the partial derivative  $\frac{1}{i} \frac{\partial}{\partial x_j}$ , we obtain a second order partial differential operator  $P(D)$  with constant coefficients. If  $\mu$  is a finite measure, then a basic formula about Fourier transforms states that

$$P(D)\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} P(x) d\mu(x).$$

Consequently, if  $\text{supp } \mu \subseteq S$ , then

$$P(D)\widehat{\mu} \equiv 0,$$

and thus  $\widehat{\mu}$  is a (distributional) solution of this PDE, as was observed in [HMR]. Conversely, if  $P(D)\widehat{\mu} \equiv 0$ , then  $\text{supp } \mu \subseteq S$ . Let us give an example of the kind of results we can obtain:

**Example 1.** Let  $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$  be the standard Laplacian on  $\mathbb{R}^d$ . The simplest bounded eigenfunctions of  $\Delta$  with eigenvalue  $\rho < 0$  are given by  $e_\lambda(x) = e^{i\langle \lambda, x \rangle}$  with  $|\lambda|^2 = -\rho$ .

Now assume that  $|\lambda|^2 = |\mu|^2 = -\rho$  and that  $e_\lambda(x) = e_\mu(x)$  on some hyperplane  $H$ . Without loss of generality, we may assume that  $H = \mathbb{R}^{d-1} \times \{0\}$  and then  $\lambda_j = \mu_j$  for  $j = 1, \dots, d-1$ . The condition  $|\lambda|^2 = |\mu|^2$  then shows that there are still two possibilities, namely  $\mu_d = \pm \lambda_d$ . Uniqueness is guaranteed as soon as we assume that  $e_\lambda(x) = e_\mu(x)$  on a second hyperplane  $H'$  that is not parallel to  $H$ .

With  $P(x) = -|x|^2 - \rho$ , the eigenvalue equation  $\Delta u = \rho u$  can be written as  $P(D)u = 0$ . Since  $e_\lambda = \widehat{\delta_\lambda}$  and  $P(\lambda) = 0$ ,  $\delta_\lambda$  is indeed supported on  $S = \{x : P(x) = 0\}$ . Theorem 1.2 then states that, for a generic pair of intersecting hyperplanes  $(H, H')$ ,  $e_\lambda(x) = e_\mu(x)$  on  $H \cup H'$  implies  $\lambda = \mu$ .

Of course, the result for the simple eigenfunctions  $e_\lambda$  is valid for an arbitrary pair of intersecting hyperplanes. However, for more general eigenfunctions, this is no longer true and some restrictions apply to the hyperplanes for the result to be true.

The actual result is much more general: Theorem 1.2 directly yields the following property of solutions of homogeneous second order PDEs.

**Theorem 1.4.** *Let  $Q$  be a quadratic form on  $\mathbb{R}^d$ ,  $v \in \mathbb{R}^d$ ,  $\rho \in \mathbb{R}$  and  $P(x) = Q(x) + \langle v, x \rangle - \rho$ . Let  $u_1, u_2 \in \mathbb{S}^{d-1}$  be such that  $Q(u_1), Q(u_2) \neq 0$  and  $u_1, u_2 \notin \mathcal{E}(Q, v, \rho)$ . Let  $\mu$  be a finite measure and assume that  $u = \hat{\mu}$  solves the partial differential equation  $P(D)u = 0$ .*

*If  $u$  vanishes on the hyperplanes  $H_{u_1}$  and  $H_{u_2}$ , then  $u = 0$ .*

In other words, a non-trivial eigenfunction of a second-order linear partial differential operator with constant coefficients cannot vanish on two generic hyperplanes. It remains to be seen whether a similar result holds for distributions instead of measures.

## 2. QUADRATIC HYPERSURFACES AND HYPERPLANES

### 2.1. Measures on quadratic hypersurfaces with Fourier transform vanishing on a hyperplane.

In this section we build up the proof of Theorem 1.2 and first investigate measures on a quadratic hypersurface  $S$  whose Fourier transform vanishes on a single hyperplane. We start with some simple geometric observations.

**Lemma 2.1.** *Let  $B$  be a bilinear form on  $\mathbb{R}^d \times \mathbb{R}^d$  and  $Q$  be the associated quadratic form  $Q(x) = B(x, x)$ . Let  $u, v \in \mathbb{R}^d$  with  $|u| = 1$  and let  $\pi_u$  be the orthogonal projection on the hyperplane  $H_u$ . Define the affine linear transformation  $R_{u,v}^Q$  by*

$$(2.2) \quad R_{u,v}^Q(x) := x - 2 \frac{B(x, u) + \langle v, u \rangle}{Q(u)} u.$$

*Fix  $x \in \mathbb{R}^d$  and consider the equation in  $y$*

$$(2.3) \quad \begin{cases} Q(y) + 2\langle y, v \rangle &= Q(x) + 2\langle x, v \rangle \\ \pi_u(y) &= \pi_u(x) \end{cases}.$$

- *If  $Q(u) \neq 0$ , then (2.3) has two solutions,  $y = x$  and  $y = R_{u,v}^Q(x)$ .*
- *If  $Q(u) = 0$  and  $B(x, u) + \langle v, u \rangle \neq 0$ , then  $y = x$  is the only solution of (2.3).*

**Notation.** If no confusion can occur, we will simply write  $R_{u,v} = R_{u,v}^Q$ . Further, if  $v = 0$  we will write  $R_u = R_{u,0}$ . In this case,  $R_u$  is a linear mapping and we define  $R_u^*$  to be its adjoint:  $\langle R_u x, y \rangle = \langle x, R_u^* y \rangle$  for every  $x, y \in \mathbb{R}^d$ .

Note that  $(R_{u,v}^Q)^2 = \text{Id}$  and that for the case  $B(x, u) = \langle x, u \rangle$  and  $v = 0$ ,  $R$  is just the reflection of a vector  $x$  across the hyperplane  $H_u$ . In general,  $R$  is a reflection across the hyperplane  $H_u$  with respect to the quadratic form  $Q(x)$  as a “metric.”



*Proof.* If  $\pi_u(x) = \pi_u(y)$  then  $y = x + tu$  for some  $t \in \mathbb{R}$ , and thus

$$Q(y) = Q(x) + t^2 Q(u) + 2tB(x, u) \quad \text{and} \quad \langle y, v \rangle = \langle x, v \rangle + t\langle u, v \rangle.$$

The identity  $Q(y) + 2\langle y, v \rangle = Q(x) + 2\langle x, v \rangle$  then reads as

$$t^2 Q(u) + 2tB(x, u) + 2t\langle u, v \rangle = 0.$$

It remains to solve for  $t$ . If  $Q(u) \neq 0$ , there are two solutions. Either  $t = 0$  and thus  $y = x$ , or  $t = -2 \frac{B(x, u) + \langle v, u \rangle}{Q(u)}$  and thus  $y = R_{u,v}(x)$ . If  $Q(u) = 0$  and  $B(x, u) + \langle v, u \rangle \neq 0$ , then  $t = 0$ , and thus  $y = x$  is the only solution.  $\square$

A measure supported on a quadratic hypersurface whose Fourier transform vanishes on a hyperplane must possess some symmetry property. The next lemma gives a precise formulation of these symmetries and is a crucial extension of Lemma 2.1 of [JK].

**Lemma 2.2.** *Let  $B$  be a bilinear form on  $\mathbb{R}^d \times \mathbb{R}^d$  and  $Q$  be the associated quadratic form  $Q(x) = B(x, x)$ . Let  $u, v \in \mathbb{R}^d$  with  $|u| = 1$ ,  $s > 0$ ,  $\rho \in \mathbb{R}$ , and let  $S = \{x \in \mathbb{R}^d : Q(x) + 2\langle x, v \rangle = \rho\}$ . For  $\mu \in \mathcal{M}(S)$  let  $\nu_{u,s}$  be the measure  $d\nu_{u,s} = e^{is\langle x, u \rangle} d\mu$ . Then the following are equivalent:*

- (i)  $\widehat{\mu}(\xi) = 0$  for every  $\xi \in H_{u,s}$ .
- (ii)  $\pi_u * \nu_{u,s} = 0$ .

Moreover, if  $Q(u) \neq 0$ , (i)-(ii) are equivalent to the following two properties:

- (iii)  $R_{u,v} * \nu_{u,s} = -\nu_{u,s}$ .
- (iv)  $\widehat{\mu}(\xi + su) = -e^{-2i \frac{\langle u, v \rangle \langle u, \xi \rangle}{Q(u)}} \widehat{\mu}(R_u^* \xi + su)$  for every  $\xi \in \mathbb{R}^d$ .

*Proof.* (i)  $\Leftrightarrow$  (ii) If  $\xi \in H_{u,s}$ , then  $\xi = \pi_u \xi + su$ , and thus

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} d\mu(x) = \int_{\mathbb{R}^d} e^{i\langle \pi_u x, \pi_u \xi \rangle} e^{is\langle x, u \rangle} d\mu(x).$$

Therefore, if we denote by  $\nu$  the measure defined by

$$d\nu(x) = d\nu_{u,s}(x) = e^{is\langle x, u \rangle} d\mu(x),$$

then, according to (1.1),

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\langle \pi_u x, \pi_u \xi \rangle} d\nu(x) = \int_{H_u} e^{i\langle y, \pi_u \xi \rangle} d\pi_u * \nu(y) = \widehat{\pi_u * \nu}(\pi_u \xi)$$

where  $\widehat{\pi_u * \nu}$  is the characteristic function of the measure  $\pi_u * \nu$  defined on  $H_u$  (identified with  $\mathbb{R}^{d-1}$ ).

Since  $\pi_u$  is a bijection from  $H_{u,s}$  to  $H_u$ ,  $\widehat{\mu} = 0$  on  $H_{u,s}$  if and only if  $\widehat{\pi_u * \nu} = 0$  on  $H_u$  that is  $\pi_u * \nu = 0$ . This establishes the equivalence of (i) and (ii).

(ii)  $\Rightarrow$  (iii) Let  $E_u = \{x \in S : B(x, u) = -\langle v, u \rangle\}$ . If  $x \in E_u$ , then  $R_{u,v}(x) = x$ , consequently  $\pi_u$  is one-to-one on  $E_u$  and  $\nu|_{E_u} = R_{u,v} * \nu|_{E_u}$ . On the complement  $S \setminus E_u$  the projection  $\pi_u$  is two-to-one. We can therefore partition  $S \setminus E_u = S_+ \cup S_-$  into two Borel sets  $S_+, S_-$  in such a way that  $\pi_u$  is one-to-one on  $S_+$  and on  $S_-$ .

Now assume that  $\pi_u * \nu = 0$ . Then  $\nu|_{E_u} = 0$ , which we may write as  $\nu|_{E_u} = -R_{u,v} * \nu|_{E_u}$ . Then

$$\begin{aligned}
 0 &= \int_{H_u} g(y) d\pi_u * \nu = \int_S g(\pi_u(x)) d\nu(x) \\
 &= \int_{E_u} g(\pi_u(x)) d\nu(x) + \int_{S_+} g(\pi_u(x)) d\nu(x) + \int_{S_-} g(\pi_u(x)) d\nu(x) \\
 (2.4) \quad &= \int_{S_+} g(\pi_u(x)) d\nu(x) + \int_{S_-} g(\pi_u(x)) d\nu(x).
 \end{aligned}$$

Since  $Q(u) \neq 0$ , Lemma 2.1 asserts that if  $x \in S_\pm$  then  $R_{u,v}(x) \in S_\mp$  and  $\pi_u(x) = \pi_u(R_{u,v}(x))$ , and  $R_{u,v}^2(x) := R_{u,v}(R_{u,v}(x)) = x$ . But then

$$\begin{aligned}
 \int_{S_-} g(\pi_u(x)) d\nu(x) &= \int_{S_-} g(\pi_u(R_{u,v}^2(x))) d\nu(x) \\
 &= \int_{S_+} g(\pi_u(R_{u,v}(x))) dR_{u,v} * \nu(x) \\
 &= \int_{S_+} g(\pi_u(x)) dR_{u,v} * \nu(x).
 \end{aligned}$$

Therefore (2.4) reads as

$$(2.5) \quad \int_{S_+} g(\pi_u(x)) d[\nu(x) + R_{u,v} * \nu(x)] = 0.$$

Since every (continuous) function on  $S_+$  can be written in the form  $g(\pi_u(x))$  for some (continuous) function  $g$  on  $H_u$ , we get  $\nu_{u,s} = -R_{u,v} * \nu_{u,s}$  on  $S_+$ . Replacing  $S_+$  by  $S_-$  in the argument shows that  $\nu_{u,s} = -R_{u,v} * \nu_{u,s}$  on  $S_-$  as well.

(iii)  $\Rightarrow$  (ii) Conversely, assume that  $\nu_{u,s} + R_{u,v} * \nu_{u,s} = 0$ . Since  $\nu_{E_u} = R_{u,v} * \nu|_{E_u}$  by symmetry, it follows that  $\nu_{u,s}|_{E_u} = 0$ , consequently  $\pi_u * \nu|_{E_u} = 0$ . Reading (2.5) and (2.4) backwards, we find that also  $\int_{S \setminus E_u} g(\pi_u(x)) d\nu(x) = 0$  and thus  $\pi_u * \nu = 0$ .

(iii)  $\Leftrightarrow$  (iv) Finally we note that

$$\widehat{\mu}(\xi + su) = \int_S e^{i\langle x, \xi + su \rangle} d\mu(x) = \int_S e^{i\langle x, \xi \rangle} d\nu(x) = \widehat{\nu}(\xi),$$

whereas

$$\begin{aligned}
 \int_S e^{i\langle x, \xi \rangle} dR_{u,v} * \nu(x) &= \int_S e^{i\langle R_{u,v}x, \xi \rangle} d\nu(x) \\
 &= e^{-2i \frac{\langle u, v \rangle \langle u, \xi \rangle}{Q(u)}} \int_S e^{i\langle R_u x, \xi \rangle} d\nu(x) \\
 &= e^{-2i \frac{\langle u, v \rangle \langle u, \xi \rangle}{Q(u)}} \int_S e^{i\langle x, R_u^* \xi + su \rangle} d\mu(x) \\
 &= e^{-2i \frac{\langle u, v \rangle \langle u, \xi \rangle}{Q(u)}} \widehat{\mu}(R_u^* \xi + su).
 \end{aligned}$$



It follows that (iii) and (iv) are equivalent.  $\square$

**Corollary 2.3.** *Let  $B$  be a bilinear form on  $\mathbb{R}^d \times \mathbb{R}^d$  and  $Q$  be the associated quadratic form  $Q(x) = B(x, x)$ . Let  $u, v \in \mathbb{R}^d$  with  $|u| = 1$ ,  $s > 0$ ,  $\rho \in \mathbb{R}$ , and let  $S = \{x \in \mathbb{R}^d : Q(x) + 2\langle x, v \rangle = \rho\}$ .*

- (i)  *$(S, H_{u,s})$  is a Heisenberg uniqueness pair if and only if  $\pi_u$  is one-to-one on  $S$ .*
- (ii) *If  $Q(u) \neq 0$ ,  $(S, H_{u,s})$  is not a Heisenberg uniqueness pair.*
- (iii) *Assume that  $Q(u) = 0$  and  $\mu \in \mathcal{M}(S)$ . Then  $\hat{\mu}$  vanishes on  $H_{u,s}$ , if and only if  $\mu$  is supported on the set*

$$E_u = \{x \in S : B(x, u) = -\langle v, u \rangle\}$$

*and  $\int_{E_u} e^{is\langle x, u \rangle} d\mu(x) = 0$ . In particular, if  $E_u \cap S$  has measure zero with respect to the surface measure on  $S$ , then  $(S, H_{u,s})$  is an  $\mathcal{AC}(S)$ -Heisenberg uniqueness pair.*

*Proof.* (i) Assume  $\pi_u$  is one-to-one on  $S$  and  $\mu \in \mathcal{M}(S)$ . Lemma 2.2 shows that if  $\hat{\mu} = 0$  on  $H_{u,s}$  then  $\pi_u * \nu_{u,s} = 0$  with  $d\nu_{u,s}(x) = e^{is\langle x, u \rangle} d\mu(x)$ . Since  $\pi_u$  is one-to-one, it follows that  $\nu_{u,s} = 0$ , thus  $\mu = 0$  and  $(S, H_{u,s})$  is a Heisenberg uniqueness pair.

Conversely, if  $\pi_u$  is not one-to-one on  $S$ , there exist points  $x, y \in S$ ,  $x \neq y$  with  $\pi_u(x) = \pi_u(y)$ . Let  $\mu = e^{-is\langle x, u \rangle} \delta_x - e^{-is\langle y, u \rangle} \delta_y \neq 0$ , so that  $\nu_{u,s} = \delta_x - \delta_y$ . Therefore  $\pi_u * \nu_{u,s} = 0$  and Lemma 2.2 shows that  $\hat{\mu} = 0$  on  $H_{u,s}$ .

(ii) If  $Q(u) \neq 0$  and  $x \in S$  satisfies  $B(x, u) + \langle v, u \rangle \neq 0$ , then Lemma 2.1 shows that  $y = R_{u,v}x \in S \setminus \{x\}$  and  $\pi_u(x) = \pi_u(y)$ . The previous argument shows that  $(S, H_{u,s})$  is not a Heisenberg uniqueness pair. The proof easily adapts to show that  $(S, H_{u,s})$  is not an  $\mathcal{AC}(S)$ -Heisenberg uniqueness pair.

(iii) If  $Q(u) = 0$ , then  $\pi_u : S \rightarrow H_u$  is one-to-one on  $S \setminus E_u$  by Lemma 2.1. Therefore if  $\pi_u * \nu_{u,s} = 0$  then  $\nu_{u,s} = 0$  on  $S \setminus E_u$  and thus  $\mu = 0$  on  $S \setminus E_u$ .  $\square$

As a single hyperplane leads to a Heisenberg uniqueness pair only in exceptional cases, we are led to ask when  $(S, H_{u_1, s_1} \cup H_{u_2, s_2} \cup \dots \cup H_{u_N, s_N})$  for  $N \geq 2$  can be a Heisenberg uniqueness pair. In this case, the measure  $\mu$  must satisfy multiple symmetries, and those symmetries may be incompatible. Our aim is to show that this is indeed the case. To do so, we will first investigate measures on  $S$  that have Fourier transform vanishing on two intersecting hyperplanes.

## 2.2. Proof of Theorem 1.2. Measures on quadratic surfaces with characteristic function vanishing on two hyperplanes.

We now prove Theorem 1.2. Throughout this section,  $Q$ ,  $B$  and  $S$  will be as above. We will prove that, unless  $(u_1, u_2)$  is in an exceptional set  $\mathcal{E}$  of measure 0, a measure  $\mu \in \mathcal{M}(S)$  such that  $\hat{\mu} = 0$  on  $H_{u_1} \cup H_{u_2}$  is necessarily  $\mu = 0$ .

We start with an appropriate parametrization of  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ . For given  $u_1 \in \mathbb{S}^{d-1}$ , every  $u_2 \in \mathbb{S}^{d-1}$  can be written as  $u_2 = \cos \theta u_1 + \sin \theta v_2$  with

unique  $v_2 \in \mathbb{S}^{d-1} \cap \{u_1\}^\perp$  (a  $d-2$  dimensional sphere) and  $\theta \in [0, \pi]$ . Note that  $\text{span}(u_1, u_2) = \text{span}(u_1, v_2)$ .

Then the Lebesgue measure on  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  decomposes as

$$\begin{aligned} & \int_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} \varphi(u_1, u_2) d\sigma_d(u_1) d\sigma_d(u_2) \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1} \cap \{u_1\}^\perp} \int_0^\pi \varphi(u_1, \cos \theta u_1 + \sin \theta v_2) \sin^{d-2} \theta d\theta d\sigma_{d-1}(v_2) d\sigma_d(u_1). \end{aligned}$$

Indeed, this is just a way to write  $d\sigma_d(u_2)$  in spherical coordinates.

To see that the exceptional set  $\mathcal{E}$  has measure zero, we will show that, for  $u_1, v_2$  fixed, the set of angles  $\theta$  for which  $(u_1, \cos \theta u_1 + \sin \theta v_2) \in \mathcal{E}$  is countable.

Note also that  $S_Q := \{u \in \mathbb{S}^{d-1} : Q(u) = 0\}$  is a lower dimensional submanifold of  $\mathbb{S}^{d-1}$  and has therefore measure 0 in  $\mathbb{S}^{d-1}$ , and so does  $S_Q \times S_Q$  in  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ .

From now on, we fix  $u_1 \in \mathbb{S}^{d-1}$ ,  $v_2 \in \mathbb{S}^{d-1} \cap \{u_1\}^\perp$  and write  $u_2 = u_2(\theta) = \cos \theta u_1 + \sin \theta v_2$ . We further assume that  $u_1, u_2 \notin S_Q$  and  $u_1 \neq u_2$ . Our aim is to show that there is an at most countable set of  $\theta$ 's for which  $(S, H_{u_1} \cup H_{u_2})$  is not an Heisenberg uniqueness pair.

**Step 1.** Let  $\mu \in \mathcal{M}(S)$  be such that  $\hat{\mu} = 0$  on  $H_{u_1} \cup H_{u_2}$ . We will write  $R_j = R_{u_j}^Q$  for the involutions on  $S$  defined by  $u_1$  and  $u_2$ . Then the equivalence (i)  $\Leftrightarrow$  (iii) of Lemma 2.2 implies that

$$R_j * \mu = -\mu, \quad j = 1, 2 \quad \text{thus} \quad (R_2 \circ R_1) * \mu = \mu.$$

Let  $\pi$  be the orthogonal projection on the subspace  $\text{span}(u_1, u_2)$  generated by  $u_1, u_2$  and  $\pi^\perp = I - \pi$  be the orthogonal projection on the orthogonal complement of  $\text{span}(u_1, u_2)$ . Let  $S_0$  be a measurable subset of  $S$  such that  $\pi^\perp$  is one-to-one from  $S_0$  to  $\text{span}(u_1, u_2)^\perp$ . Consequently, we may write every  $y \in S$  as  $y = x + u$  for a unique  $x \in S_0$  and  $u \in \text{span}(u_1, u_2)$ .

We now make two crucial observations: (i) The intersection of  $S$  with the affine plane  $x + \text{span}(u_1, u_2)$  is a conic section (an ellipse, parabola, hyperbola, or two lines), and (ii) every such cross section  $S \cap (x + \text{span}(u_1, u_2))$  is invariant under the involutions  $R_1$  and  $R_2$ . This follows from Lemma 2.1.

We may therefore use the analysis of Theorem 1.1 for each section and each pair of lines  $x + \mathbb{R}u_1$  and  $x + \mathbb{R}u_2$ . To apply this analysis, we further need an appropriate restriction of the given measure  $\mu$  to the cross section  $S \cap (x + \text{span}(u_1, u_2))$ . Technically, this is done by disintegration.

Let  $\nu = \pi^\perp * \mu$  and let us write the disintegration theorem (e.g., [DM]) in the form

$$(2.6) \quad \int_S \varphi(x) d\mu(x) = \int_{S_0} \int_{(\pi^\perp)^{-1}(x)} \varphi(y) d\mu_x(y) d\nu(x) \quad \varphi \in \mathcal{C}_c(S).$$

Recall that the measure  $\mu_x$  is uniquely determined by this formula for  $\nu$ -almost every  $x$  and that  $\mu_x$  is supported on  $S \cap (x + \text{span}(u_1, u_2))$ .

Since  $\pi^\perp u_j = 0$  for  $j = 1, 2$ , (2.2) implies that  $\pi^\perp R_j = \pi^\perp$ . Therefore, for  $\varphi \in \mathcal{C}_c(S)$ , we have

$$\begin{aligned}
 \int_S \varphi(x) dR_j * \mu(x) &= \int_S \varphi(R_j(x)) d\mu(x) \\
 &= \int_{S_0} \int_{(\pi^\perp)^{-1}(x)} \varphi(R_j(y)) d\mu_x(y) d\nu(x) \\
 (2.7) \quad &= \int_{S_0} \int_{(\pi^\perp)^{-1}(x)} \varphi(y) dR_j * \mu_x(y) d\nu(x).
 \end{aligned}$$

It follows that  $R_j * \mu = -\mu$  if and only if  $R_j * \mu_x = -\mu_x$  for  $\nu$ -almost every  $x$ . The equivalence (i)  $\Leftrightarrow$  (iii) of Lemma 2.2 implies that  $\widehat{\mu_x} = 0$  on  $\mathbb{R}u_1 \cup \mathbb{R}u_2$ .<sup>‡</sup>

We may summarize this reduction to the two-dimensional case as follows.

**Lemma 2.4.** *Let  $\mu \in \mathcal{M}(S)$  and  $u_1, u_2 \in \mathbb{S}^{d-1}$ . Then  $\widehat{\mu}$  vanishes on  $H_{u_1} \cup H_{u_2}$ , if and only if  $\widehat{\mu_x}$  vanishes on  $\mathbb{R}u_1 \cup \mathbb{R}u_2$  for  $\nu$ -almost all  $x \in S_0$ .*

Next, for  $x \in S_0$  the measure  $\mu_x$  is supported on the intersection of the quadratic surface  $S$  with the plane  $x + \text{span}(u_1, u_2) = x + \text{span}(u_1, v_2)$ . This intersection is either the union of two lines, a parabola, a hyperbola or an ellipse. For each of these, we know from Theorem 1.1 when the condition  $\widehat{\mu_x}|_{\mathbb{R}u_1 \cup \mathbb{R}u_2} = 0$  implies that  $\mu_x = 0$ .

**Step 2.** We now apply Theorem 1.1 to  $S \cap (x + \text{span}(u_1, u_2))$  and  $\mu_x$ . We first determine the intersection  $S_x = S \cap (x + \text{span}(u_1, u_2)) = S \cap (x + \text{span}(u_1, v_2))$  precisely.

As  $x \in S$ , a point  $x + su_1 + tv_2$  belongs to  $S$ , if and only if

$$Q(x + su_1 + tv_2) + 2\langle x + su_1 + tv_2, v \rangle = Q(x) + 2\langle x, v \rangle = \rho,$$

or equivalently,

$$(2.8) \quad s^2 Q(u_1) + 2stB(u_1, v_2) + t^2 Q(v_2) + 2sb(x) + 2tb'(x) = 0.$$

with  $b(x) = B(x, u_1) + \langle u_1, v \rangle$  and  $b'(x) = B(x, v_2) + \langle v_2, v \rangle$ .

Let  $\Sigma$  be the set of points  $(s, t) \in \mathbb{R}^2$  satisfying (2.8). Then  $\Sigma$  is either a hyperbola, an ellipse, or a parabola in  $\mathbb{R}^2$  (where a degenerate hyperbola is a set of two intersecting lines and a degenerate ellipse is a point or the empty set). The classification into these three types depends on the value of the discriminant of the quadratic form  $\tilde{Q}(s, t) = s^2 Q(u_1) + 2stB(u_1, v_2) + t^2 Q(v_2)$ , namely  $\Delta = \det \begin{pmatrix} Q(u_1) & B(u_1, v_2) \\ B(u_1, v_2) & Q(v_2) \end{pmatrix} = Q(u_1)Q(v_2) - B(u_1, v_2)^2$ . The set

$$S_x = S \cap (x + \text{span}(u_1, v_2)) = \{x + su_1 + tv_2 : (s, t) \in \Sigma\} \subseteq \mathbb{R}^d$$

is obtained from  $\Sigma$  by a linear transformation and a shift by  $x$  and thus represents the same conic section as  $\Sigma$ . Specifically, if  $Q(u_1)Q(v_2) - B(u_1, v_2)^2 < 0$ ,

<sup>‡</sup>The Fourier transform of  $\mu_x$  is here taken in  $x + \text{span}(u_1, u_2)$ . That is

$$\widehat{\mu_x}(\xi u_1 + \eta u_2) = \int_{\mathbb{R}^2} e^{i(s\xi + t\eta)} d\mu_x(x + su_1 + tv_2).$$

then, for all  $x \in S_0$ ,  $S \cap (x + \text{span}(u_1, v_2))$  is a hyperbola or two intersecting lines (in the degenerate case). If  $Q(u_1)Q(v_2) - B(u_1, v_2)^2 > 0$ , then  $S \cap (x + \text{span}(u_1, v_2))$  is an ellipse or a point (in the degenerate case). If  $Q(u_1)Q(v_2) - B(u_1, v_2)^2 = 0$ , then  $S \cap (x + \text{span}(u_1, v_2))$  is a parabola or a set of two parallel lines or a single line (in the degenerate case).

Theorem 1.1 states the existence of a set  $\mathcal{E}_x = \mathcal{E}(\tilde{Q}) \subseteq \text{span}(u_1, v_2)$  of pairs of directions such that if  $(w_1, w_2) \notin \mathcal{E}_x$  then  $\mu_x = 0$ . At this point it is crucial that  $\mathcal{E}_x$  depends only on  $\tilde{Q}$ , but not on  $x \in S_0$ .

Consequently, if  $w_1, w_2 \notin \mathcal{E}(\tilde{Q})$ , then  $\mu_x = 0$  for all  $x \in S_0$ . By (2.6) this implies that  $\mu = 0$ .

**Step 3.** We finally calculate the measure of  $\mathcal{E}$  in  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ . Fix  $u_1$  and  $v_2 \in u_1^\perp$ , and take  $w_1 = u_1$  and  $w_2 = \cos \theta u_1 + \sin \theta v_2$ . Then the set of  $\theta$ , such that  $(w_1, w_2) \in \mathcal{E}_x$  is contained in a fixed countable set independent of  $x$ . Consequently the measure of the exceptional set  $\mathcal{E}$  is

$$(2.9) \quad \int_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} \chi_{\mathcal{E}}(u_1, u_2) d\sigma_d(u_1) d\sigma_d(u_2) \\ = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1} \cap \{u_1\}^\perp} \int_0^\pi \chi_{\mathcal{E}}(u_1, \cos \theta u_1 + \sin \theta v_2) \sin^{d-2} \theta d\theta d\sigma_{d-1}(v_2) d\sigma_d(u_1) = 0.$$

This completes the proof of Theorem 1.2.

### 3. THE CONE AND THE HYPERBOLOID

In this section we investigate special arrangements of hyperplanes for  $S$  being a hyperboloid or a cone.

Let  $p$  be an integer,  $1 \leq p \leq d-1$  and  $q = d-p$ . Let  $Q$  be the quadratic form

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

with the associated bilinear form  $B$  and set  $v = 0$ . Let  $\rho \in \mathbb{R}$  and  $S_\rho = \{x \in \mathbb{R}^d : Q(x) = \rho\}$ . Then  $S_0$  is a cone and  $S_\rho, \rho \neq 0$ , is a hyperboloid with one or two connected components. For instance, in dimension 3, if  $p = 1$  and  $q = 2$  then  $S_1$  has 2 sheets while  $S_{-1}$  has only one.

Our aim is to complement the statement of Theorem 1.2 which treats the case of pairs of intersecting hyperplanes  $H_{u_1} \cup H_{u_2}$  with non-isotropic normals,  $Q(u_1), Q(u_2) \neq 0$ . We will treat two cases. On one hand, we will consider the case of parallel hyperplanes and show that, generically,  $(S, H_{u, s_1} \cup H_{u, s_2} \cup H_{u, s_3})$  is a Heisenberg uniqueness pair. On the other hand, we will also show that it is possible to construct Heisenberg uniqueness pairs of the form  $(S, H_{u_1} \cup \cdots \cup H_{u_k})$  where all normals  $u_1, \dots, u_k$  are isotropic vectors.

In this section, we will write  $\mathbb{R}^d = \mathbb{R}^p \times \mathbb{R}^q$  in the obvious sense: if  $x \in \mathbb{R}^d$ , then  $x'$  is its projection on  $\mathbb{R}^p$  and  $x''$  its projection on  $\mathbb{R}^q$  so that  $x = (x', x'')$ . In particular,  $\langle x, u \rangle = \langle x', u' \rangle + \langle x'', u'' \rangle$  and  $B(x, u) = \langle x', u' \rangle - \langle x'', u'' \rangle$ .

Note also that, if  $|u| = 1$  and  $Q(u) \neq 0$ , then

$$\begin{aligned} R_u x &= \langle x, u \rangle u + \pi_{u^\perp} x - 2 \frac{B(x, u)}{Q(u)} u \\ &= \frac{(Q(u) - 2) \langle x', u' \rangle + (Q(u) + 2) \langle x'', u'' \rangle}{Q(u)} u + \pi_{u^\perp} x. \end{aligned}$$

Set

$$\tilde{u} = \frac{\left( (Q(u) - 2)u', (Q(u) + 2)u'' \right)}{Q(u)},$$

then  $R_u^* u = \tilde{u} = \langle \tilde{u}, u \rangle u + \pi_{u^\perp} \tilde{u}$ . Since the normalization  $|u'|^2 + |u''|^2 = 1$  implies that  $|u'|^2 = \frac{1 + Q(u)}{2}$  and  $|u''|^2 = \frac{1 - Q(u)}{2}$ , we thus obtain  $\langle \tilde{u}, u \rangle = -1$ . An easy computation now shows that

$$\begin{aligned} R_u^* \xi &= R_u^* (\langle \xi, u \rangle u + \pi_{u^\perp} \xi) \\ &= \langle \xi, u \rangle R_u^* u + \pi_{u^\perp} \xi \\ &= -\langle \xi, u \rangle u + \langle \xi, u \rangle \pi_{u^\perp} \tilde{u} + \pi_{u^\perp} \xi. \end{aligned}$$

Recall from Lemma 2.2 that, when  $Q(u) \neq 0$  and  $\mu \in \mathcal{M}(S)$ , then  $\hat{\mu} = 0$  on  $H_{u,s}$  if and only if

$$(3.10) \quad \hat{\mu}(\xi + su) = -\hat{\mu}(R_u^* \xi + su) \quad \text{for all } \xi \in \mathbb{R}^d.$$

### 3.1. Parallel hyperplanes with non-isotropic normal.

We will now prove the following statement.

**Proposition 3.1.** *Let  $1 \leq p, q \leq d - 1$  be two integers with  $p + q = d$ ,  $s_1, s_2, s_3 \in \mathbb{R}$ , and let*

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

*Let  $u \in \mathbb{R}^d$  be such that  $Q(u) \neq 0$ ,  $\rho \in \mathbb{R}$ , and set  $S_\rho = \{x \in \mathbb{R}^d : Q(x) = \rho\}$ . If  $s_1, s_2, s_3 \in \mathbb{R}$  are linearly independent over  $\mathbb{Q}$ , then  $(S_\rho, H_{u,s_1} \cup H_{u,s_2} \cup H_{u,s_3})$  is a Heisenberg uniqueness pair.*

*Proof.* Translating the hyperplanes in the direction  $u$ , it is enough to consider the case  $s_1 = 0$ . We thus want to prove that  $(S_\rho, H_{u,0} \cup H_{u,s} \cup H_{u,t})$  is a Heisenberg uniqueness pair when  $s, t$  are  $\mathbb{Q}$ -independent.

From (3.10), we obtain that  $\hat{\mu}$  satisfies the following symmetries

$$\begin{cases} \hat{\mu}(\xi) &= -\hat{\mu}(R_u^* \xi) \\ \hat{\mu}(\xi + su) &= -\hat{\mu}(R_u^* \xi + su) \\ \hat{\mu}(\xi + tu) &= -\hat{\mu}(R_u^* \xi + tu) \end{cases}$$

Now for arbitrary  $\xi \in H_{u,0} = u^\perp$  write  $\xi + s_0 u = \xi + (s_0 - s)u + su$  and note that  $R_u^*(\xi + (s_0 - s)u) = (s - s_0)u + (s_0 - s)\pi_{u^\perp} \tilde{u} + \xi$ , since  $\pi_{u^\perp} \xi = \xi$ . It follows that, on the one hand,

$$(3.11) \quad \hat{\mu}(\xi + s_0 u) = -\hat{\mu}(R_u^*(\xi + s_0 u)) = -\hat{\mu}(-s_0 u + s_0 \pi_{u^\perp} \tilde{u} + \xi),$$

and on the other hand

$$\begin{aligned} \widehat{\mu}(\xi + s_0 u) &= \widehat{\mu}(\xi + (s_0 - s)u + su) = -\widehat{\mu}(R_u^*(\xi + (s_0 - s)u) + su) \\ (3.12) \qquad \qquad \qquad &= -\widehat{\mu}(\xi + (s_0 - s)\pi_{u^\perp} \tilde{u} + (2s - s_0)u). \end{aligned}$$

Now assume that  $\widehat{\mu} = 0$  on  $H_{u,s_0}$ . Then (3.11) implies that  $\widehat{\mu} = 0$  on  $H_{u,-s_0}$ . Applying now (3.12) with  $-s_0$  instead of  $s_0$ , we obtain that  $\widehat{\mu} = 0$  on  $H_{u,s_0+2s}$ . By induction, it follows that  $\widehat{\mu} = 0$  on  $H_{u,s_0+2ks}$ , for every  $k \in \mathbb{N}$ . Reversing the order in which we use (3.11)-(3.12) we obtain that  $\widehat{\mu} = 0$  on  $H_{u,s_0+2ks}$  for every  $k \in \mathbb{Z}$ .

Next, using the third symmetry we obtain that  $\widehat{\mu} = 0$  on  $H_{u,s_0+2ks+2\ell t}$  for every  $k, \ell \in \mathbb{Z}$ . As  $\widehat{\mu} = 0$  on  $H_{u,0}$  we get that  $\widehat{\mu} = 0$  on every hyperplane  $H_{u,2ks+2\ell t}$  for every  $k, \ell \in \mathbb{Z}$ . This set is dense in  $\mathbb{R}^d$  and  $\widehat{\mu}$  is continuous, therefore  $\widehat{\mu} = 0$  and  $(S, H_{u,0} \cup H_{u,s} \cup H_{u,t})$  is a Heisenberg uniqueness pair.  $\square$

### 3.2. Intersecting hyperplanes with isotropic normal.

Again, let  $1 \leq p \leq q \leq d-1$ ,  $p+q=d$  with  $p+q=d$ ,  $Q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$ , and  $S_\rho = \{x \in \mathbb{R}^d : Q(x) = \rho\}$  for  $\rho \in \mathbb{R}$ . We now assume that  $u \in \mathbb{R}^d$  is isotropic, i.e.,  $Q(u) = 0$ . Write  $u = (u', u'') \in \mathbb{R}^p \times \mathbb{R}^q$ , then  $Q(u) = 0$  is equivalent to  $|u'| = |u''|$ . Without loss of generality, we can restrict our attention to  $u$  with  $|u'| = |u''| = 1$ .

Let  $\mu \in \mathcal{M}(S_\rho)$ . By Corollary 2.3,  $\widehat{\mu}$  vanishes on  $H_u$ , if and only if  $\mu$  is supported on the set  $E_u = \{x \in S_\rho : B(x, u) = 0\}$ . We represent  $E_u$  in a different form as follows: a point  $(x', x'')$  is in  $S_\rho$ , if and only if  $|x'|^2 - |x''|^2 = \rho$ , i.e.,  $x' \in (|x''|^2 + \rho)^{1/2} \mathbb{S}^{p-1}$ , and  $B(x, u) = 0$ , if and only if  $\langle x', u' \rangle = \langle x'', u'' \rangle$ . Consequently,

$$(3.13) \quad E_u = \{(x', x'') \in \mathbb{R}^{p+q} : x' \in (|x''|^2 + \rho)^{1/2} \mathbb{S}^{p-1} \cap (\langle x'', u'' \rangle u' + (u')^\perp)\}.$$

We note right away that  $E_u$  is symmetric, and thus  $x \in E_u$  implies that  $-x \in E_u$ .

This leads to the first observation on intersecting hyperplanes with isotropic normals.

#### Proposition 3.2.

- (i) If  $Q(u) = 0$  for non-zero  $u$ , then  $(S_\rho, H_u)$  is an  $\mathcal{AC}(S)$ -Heisenberg uniqueness pair.
- (ii) Given vectors  $u_j, j = 1, \dots, k$ , such that  $Q(u_1) = \dots = Q(u_k) = 0$ ,  $(S_\rho, H_{u_1} \cup \dots \cup H_{u_k})$  is a Heisenberg uniqueness pair, if and only if  $\bigcap_{j=1}^k E_{u_j}$  is either empty or  $\{0\}$ .
- (iii) If  $k < p$  and  $Q(u_1) = \dots = Q(u_k) = 0$ , then  $(S_\rho, H_{u_1} \cup \dots \cup H_{u_k})$  is not a Heisenberg uniqueness pair.

*Proof.* (i) Since  $E_u$  is a  $(d-2)$ -dimensional submanifold of the  $(d-1)$ -dimensional manifold  $S_\rho$ , it has surface measure zero in  $S_\rho$ . An absolutely continuous measure with support in  $E_u$  is necessarily 0 (Corollary 2.3(iii)).

(ii) Next assume that  $u_1, \dots, u_k$  are such that  $Q(u_1) = \dots = Q(u_k) = 0$ . By Corollary 2.3,  $\widehat{\mu}$  vanishes on  $\bigcup_{j=1}^k H_{u_j}$ , if and only if  $\mu$  is supported

on  $\bigcap_{j=1}^k E_{u_j}$ . If this intersection is empty, then obviously  $\mu = 0$ . If this intersection is  $\{0\}$ , then  $\mu = c\delta_0$ . As  $\widehat{\mu}(0) = 0$ , we obtain  $\mu = 0$ . Note that if  $\rho \neq 0$ , then  $0 \notin S_\rho$  and we obtain directly that  $\mu = 0$ .

If  $\bigcap_{j=1}^k E_{u_j}$  contains at least two points, then there is a non-zero  $x$  such that  $\{x, -x\} \subset \bigcap_{j=1}^k E_{u_j}$  by the symmetry of  $E_u$ . Now set  $\mu = \delta_x - \delta_{-x} \in \mathcal{M}(S)$ . Then  $\mu$  is non-zero, but  $\widehat{\mu} = 0$  on  $H_{u_1} \cup \dots \cup H_{u_k}$ , so  $(S_\rho, H_{u_1} \cup \dots \cup H_{u_k})$  is not a Heisenberg uniqueness pair.

(iii) If  $k < p$ , then there exists a non-zero  $x'' \in (u_1'')^\perp \cap \dots \cap (u_k'')^\perp$  of norm  $|x''|^2 > |\rho|$  and a  $x' \in (|x''|^2 + \rho)^{1/2} \mathbb{S}^{p-1} \cap \bigcap_{j=1}^k (u_j')^\perp$ . Consequently  $E_{u_1} \cap \dots \cap E_{u_k}$  is not empty and contains the non-zero point  $(x', x'')$ . By item (ii)  $(S_\rho, H_{u_1} \cup \dots \cup H_{u_k})$  is not a Heisenberg uniqueness pair.  $\square$

Using the geometric characterization (ii) of Proposition 3.2, one can now derive numerous examples of Heisenberg uniqueness pairs. As an example we prove the following statements.

**Proposition 3.3.** *Let  $\rho \in \mathbb{R}$ ,  $1 \leq p \leq q \leq d-1$  with  $p+q = d$ ,  $Q$  be a quadratic form*

$$Q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

and  $S_\rho$  be the cone ( $\rho = 0$ ) or hyperboloid ( $\rho \neq 0$ )  $S_\rho = \{x \in \mathbb{R}^d : Q(x) = \rho\}$ .

- (i) *If  $\rho > 0$ , there exist  $u_1, \dots, u_p \neq 0$  with  $Q(u_1) = \dots = Q(u_p) = 0$ , such that  $(S_0, H_{u_1} \cup \dots \cup H_{u_p})$  is a Heisenberg uniqueness pair.*
- (ii) *If  $\rho = 0$ , there exist  $2p$  vectors  $u_1, \dots, u_{2p} \neq 0$  with  $Q(u_1) = \dots = Q(u_{2p}) = 0$  such that  $(S_0, H_{u_1} \cup \dots \cup H_{u_{2p}})$  is a Heisenberg uniqueness pair.*
- (iii) *If  $\rho < 0$ , there exist  $p+q$  vectors  $u_1, \dots, u_{p+q} \neq 0$  with  $Q(u_1) = \dots = Q(u_{p+q}) = 0$  such that  $(S_0, H_{u_1} \cup \dots \cup H_{u_{p+q}})$  is a Heisenberg uniqueness pair.*

*Proof.* Choose an orthonormal basis  $u_1', \dots, u_p'$  of  $\mathbb{R}^p$ . Define  $u_j'' = (u_j', 0, \dots, 0) \in \mathbb{R}^q$ ,  $u_j = (u_j', u_j'')$ , and  $\tilde{u}_j = (u_j', -u_j'')$ ,  $j = 1, \dots, p$ .

First note that, for every  $a_1, \dots, a_p$ , the intersection

$$(a_1 u_1' + (u_1')^\perp) \cap \dots \cap (a_p u_p' + (u_p')^\perp)$$

contains exactly one point, namely,  $x' = \sum_{j=1}^p a_j u_j'$ .

If  $(x', x'') \in \bigcap_{j=1, \dots, p} E_{u_j}$  then, in view of (3.13),  $x' \in \bigcap_{j=1}^p (\langle x'', u_j'' \rangle u_j' + (u_j')^\perp)$ , so that

$$x' = \sum_{j=1}^p \langle x'', u_j'' \rangle u_j' \quad \text{and} \quad |x'|^2 = \sum_{j=1}^p \langle x'', u_j'' \rangle^2.$$

In particular, if we write  $x'' = (y, \tilde{y})$  with  $y \in \mathbb{R}^p$  and  $\tilde{y} \in \mathbb{R}^{q-p}$  (with the obvious abuse of notation when  $q = p$ ), then  $x' = y$ .



Let us first assume that  $\rho > 0$  then  $|x'| \leq |x''| < (|x''|^2 + \rho)^{1/2}$ . This contradicts (3.13) and thus implies that  $\bigcap_{j=1,\dots,p} E_{u_j} = \emptyset$ . Applying Proposition 3.2,

this proves (i).

If  $\rho \leq 0$ ,

$$\bigcap_{j=1,\dots,p} E_{u_j} = \{(x', x', \tilde{x}), x' \in \mathbb{R}^p, \tilde{x} \in \mathbb{R}^{q-p}\},$$

and similarly,

$$\bigcap_{j=1,\dots,p} E_{\tilde{u}_j} = \{(x', -x', \tilde{x}), x' \in \mathbb{R}^p, \tilde{x} \in \mathbb{R}^{q-p}\}.$$

If  $q = p$ , then already  $\bigcap_{j=1,\dots,p} E_{u_j} \cap \bigcap_{j=1,\dots,p} E_{\tilde{u}_j} = \{0\}$ .

For  $\rho = 0$  the condition  $|x'| = |x''| = |(x', \tilde{x})|$  implies that  $\tilde{x} = 0$ , and again  $\bigcap_{j=1,\dots,p} E_{u_j} \cap \bigcap_{j=1,\dots,p} E_{\tilde{u}_j} = \{0\}$ .

In both cases, Proposition 3.2 shows that  $(S_0, H_{u_1} \cup \dots \cup H_{u_p} \cup H_{\tilde{u}_1} \cup \dots \cup H_{\tilde{u}_p})$  is a Heisenberg uniqueness pair. Thus (ii) is proved.

Now, if  $\rho < 0$  and  $q > p$ , then

$$\bigcap_{j=1,\dots,p} E_{u_j} \cap \bigcap_{j=1,\dots,p} E_{\tilde{u}_j} = \{(0, \dots, 0, \tilde{x}), \tilde{x} \in \mathbb{R}^{q-p}\}.$$

Complete the orthonormal set  $u_j'', j = 1, \dots, p$  with vectors  $u_j'', j = p+1, \dots, q$ , into an orthonormal basis of  $\mathbb{R}^q$  and let  $u_j = (u_1', u_j'')$ ,  $j = p+1, \dots, q$ . If  $x \in \bigcap_{j=1,\dots,q} E_{u_j} \cap \bigcap_{j=1,\dots,p} E_{\tilde{u}_j}$ , then on the one hand,  $x = (0, \dots, 0, \tilde{x})$  with  $\tilde{x} \in \mathbb{R}^{q-p}$ . On the other hand, since also  $x \in E_{u_j}$  for  $j = p+1, \dots, q$ , we must have  $0 \in \langle (0, \tilde{x}), u_j'' \rangle u_1' + (u_1')^\perp$ . Thus  $\langle (0, \tilde{x}), u_j'' \rangle = 0$  and finally  $\tilde{x} = 0$ . Again, from Proposition 3.2 we get that  $(S_0, H_{u_1} \cup \dots \cup H_{u_q} \cup H_{\tilde{u}_1} \cup \dots \cup H_{\tilde{u}_p})$  is a Heisenberg uniqueness pair.  $\square$

#### 4. THE SPHERE

In this section  $Q(x) = |x|^2$ ,  $B(x, y) = \langle x, y \rangle$ ,  $v = 0$  and  $S = \mathbb{S}^{d-1}$  is the unit sphere of  $\mathbb{R}^d$ . If  $|u| = 1$ ,  $R_u$  is the reflection with respect to the hyperplane normal to  $u$  and  $R_u^* = R_u$ .

Let us first consider the case of parallel hyperplanes.

**Proposition 4.1.** *Let  $u \in \mathbb{S}^{d-1}$  and  $s_1 \neq s_2 \in \mathbb{R}$ . Then*

- (i)  $(\mathbb{S}^{d-1}, H_{u,s_1} \cup H_{u,s_2})$  is an  $\mathcal{AC}(\mathbb{S}^{d-1})$ -Heisenberg uniqueness pair.
- (ii)  $(\mathbb{S}^{d-1}, H_{u,s_1} \cup H_{u,s_2})$  is a Heisenberg uniqueness pair if and only if  $|s_1 - s_2| > \frac{\pi}{2}$ .
- (iii) If  $s_1, s_2, s_3$  are linearly independent over  $\mathbb{Q}$ , then  $(\mathbb{S}^{d-1}, H_{u,s_1} \cup H_{u,s_2} \cup H_{u,s_3})$  is a Heisenberg uniqueness pair.

The proof of (i) is essentially the same as in the case  $d = 2$  by N. Lev [Le].

*Proof.* After a translation, it is enough to prove that  $(\mathbb{S}^{d-1}, H_{u,-s} \cup H_{u,s})$  is a Heisenberg uniqueness pair, where  $s = |s_1 - s_2|/2$ . After applying a suitable

rotation, we may further assume that  $u = (1, 0, \dots, 0)$ . We will use the following notation: we write  $x = (x_1, \bar{x})$  for  $x \in \mathbb{R}^d$ .

Let  $\mu \in \mathcal{M}(\mathbb{S}^{d-1})$ . According to Lemma 2.2,  $\widehat{\mu} = 0$  on  $H_{u,-s} \cup H_{u,s}$  is equivalent to  $R_{u*}(e^{\pm isx_1} d\mu(x_1, \bar{x})) = -e^{\pm isx_1} d\mu(x_1, \bar{x})$ . But  $R_{u*}(e^{\pm isx_1} d\mu(x_1, \bar{x})) = e^{\mp isx_1} d\mu(-x_1, \bar{x})$  thus

$$(4.14) \quad \begin{cases} e^{+isx_1} d\mu(x_1, \bar{x}) + e^{-isx_1} d\mu(-x_1, \bar{x}) = 0 \\ e^{-isx_1} d\mu(x_1, \bar{x}) + e^{+isx_1} d\mu(-x_1, \bar{x}) = 0 \end{cases}.$$

Writing each equation in the form  $e^{\pm 2isx_1} d\mu(x_1, \bar{x}) = -d\mu(-x_1, \bar{x})$  and subtracting these equations, we obtain that  $\sin(2sx_1) d\mu(x_1, \bar{x}) = 0$ . Consequently  $\mu$  is

supported on  $\mathbb{S}^{d-1} \cap (\frac{\pi}{2s}\mathbb{Z} \times \mathbb{R}^{d-1})$ . Further, (4.14) shows that  $\mu = 0$  on  $\mathbb{S}^{d-1} \cap (\{0\} \times \mathbb{R}^{d-1})$  so that  $\mu$  is actually supported on  $\mathbb{S}^{d-1} \cap (\frac{\pi}{2s}\mathbb{Z}^* \times \mathbb{R}^{d-1})$ .

Note that  $\mu$  is supported on a set which has surface measure zero on  $\mathbb{S}^{d-1}$ . Thus, if  $\mu$  is absolutely continuous with respect to surface measure on  $\mathbb{S}^{d-1}$ , then  $\mu = 0$  and  $(\mathbb{S}^{d-1}, H_{u,-s} \cup H_{u,s})$  is an  $\mathcal{AC}(\mathbb{S}^{d-1})$ -Heisenberg uniqueness pair.

In the general case,  $\mu = \sum_{k \in \mathbb{Z}^*} \delta_{\frac{\pi}{2s}k} \otimes \mu_k$ . Since  $\text{supp } \mu \subseteq \mathbb{S}^{d-1}$ , this representation of  $\mu$  implies that  $\mu_k = 0$  if  $\frac{\pi}{2s}|k| > 1$  and that  $\mu_k$  is supported on  $(1 - \frac{\pi^2}{4s^2}k^2)^{1/2} \mathbb{S}^{d-2}$  if  $|k| \leq \frac{2}{\pi}s$ . Moreover, the symmetry (4.14) implies that  $e^{i\frac{\pi}{2}k} \mu_k + e^{-i\frac{\pi}{2}k} \mu_{-k} = 0$ , that is  $\mu_{-k} = (-1)^{k-1} \mu_k$ . In particular, if  $s > \frac{\pi}{2}$ , then  $\mu = \delta_0 \otimes \mu_0$  and thus  $\mu = 0$ .

On the other hand, if  $s \leq \frac{\pi}{2}$ , we take the  $\mu_k$ 's to be arbitrary measures supported on  $(1 - \frac{\pi^2}{4s^2}k^2)^{1/2} \mathbb{S}^{d-2}$  for  $0 < k \leq \frac{2}{\pi}s$  and then define  $\mu_{-k} = (-1)^{k-1} \mu_k$  for those  $k$ 's. The measure  $\mu = \sum_{0 < |k| \leq \frac{2}{\pi}s} \delta_{\frac{\pi}{2s}k} \otimes \mu_k$  satisfies  $\widehat{\mu} = 0$  on  $H_{u,-s} \cup H_{u,s}$ . Thus  $(\mathbb{S}^{d-1}, H_{u,-s} \cup H_{u,s})$  is not a Heisenberg uniqueness pair.

Note that in this case,  $\widehat{\mu}(\xi_1, \bar{\xi}) = \sum_{0 < |k| \leq \frac{2}{\pi}s} e^{-\frac{i\pi}{2s}k\xi_1} \widehat{\mu_k}(\bar{\xi})$  is  $4\pi s$  periodic in the first variable. Consequently, after a translation, if  $\mu \in \mathcal{M}(\mathbb{S}^{d-1})$  and  $\widehat{\mu} = 0$  on  $H_{0,s_1} \cup H_{0,s_2}$ , then  $\widehat{\mu}$  is  $2(s_2 - s_1)\pi$  periodic in the first variable  $\xi_1$ . Thus, if  $\widehat{\mu} = 0$  on  $H_{0,s_3}$  as well, then  $\widehat{\mu}$  is also  $2(s_3 - s_1)\pi$  periodic in  $\xi_1$ . In particular, for every  $k, \ell \in \mathbb{Z}$  we have  $\widehat{\mu}(s_1 + 2k(s_2 - s_1)\pi + 2\ell(s_3 - s_1)\pi, \bar{\xi}) = \widehat{\mu}(s_1, \bar{\xi}) = 0$ . If  $s_1, s_2, s_3$  are linearly independent over  $\mathbb{Q}$  then the set  $\{s_1 + 2k(s_2 - s_1)\pi + 2\ell(s_3 - s_1)\pi : k, \ell \in \mathbb{Z}\}$  is dense in  $\mathbb{R}$  thus  $\widehat{\mu} = 0$  on a dense set in  $\mathbb{R}^d$ . As  $\widehat{\mu}$  is continuous,  $\widehat{\mu} = 0$  on  $\mathbb{R}^d$  and thus  $\mu = 0$ .  $\square$

Let us now turn to  $N$  hyperplanes  $H_1, \dots, H_N$  that intersect in a common point. Without loss of generality, we will assume that  $H_1, \dots, H_N$  intersect in 0 so that  $H_k = H_{u_k,0}$  for some  $u_k \in \mathbb{R}^d, |u_k| = 1$ .

**Theorem 4.2.** *Let  $H_1, \dots, H_N$  be hyperplanes in  $\mathbb{R}^d$  and  $R_1, \dots, R_N$  be the corresponding orthogonal reflections. Then  $(S^{d-1}, H_1 \cup \dots \cup H_N)$  is a Heisenberg uniqueness pair if and only if the Coxeter group generated by  $R_1, \dots, R_N$  is infinite.*

*Proof.* Let  $R_k$  be the reflection with respect to  $H_k$ . If  $\mu \in M(\mathbb{S}^{d-1})$  and  $\hat{\mu}$  vanishes on  $H_k$ , then by Lemma 2.2 we have  $R_k * \mu = -\mu$ . Consequently, its Fourier transform satisfies  $|\hat{\mu}(R_k \xi)| = |\hat{\mu}(\xi)|$  for all  $\xi \in \mathbb{R}^d$ .

Let  $G$  be the group generated by the reflections  $\{R_1, \dots, R_k\}$ . This is a Coxeter group. The modulus  $|\hat{\mu}|$  is therefore invariant under the Coxeter group  $G = \langle R_1, \dots, R_k \rangle$ . Let  $u, u' \in \mathbb{S}^{d-1}$  and  $x \in H_u$ . Since  $\langle R_{u'} x, R_{u'} u \rangle = \langle x, u \rangle$ , we have  $R_{u'} H_u = H_{R_{u'} u}$ . Consequently, if  $g = R_{j_1} R_{j_2} \dots R_{j_n} \in G$  and  $\hat{\mu} = 0$  on  $H_u$ , then  $\hat{\mu}$  vanishes on  $H_{gu}$ . Furthermore note that the composition  $R_{u'} R_u$  of two reflections is a rotation in the plane spanned by  $u$  and  $u'$  with angle twice the angle between  $u$  and  $u'$ .

We distinguish two cases:

(i) Either  $G$  is infinite. In this case [HLR, Lemma 4.9],  $G$  contains a subgroup generated by two reflections  $R, R'$  that is already infinite. Write  $H_u, H_{u'}$  for the corresponding hyperplanes and note that  $u' \notin H_u$ . But then the rotation  $RR'$  has an angle that is an irrational multiple of  $\pi$ , so that the orbit of  $H_u$  under  $RR'$  is dense in  $\mathbb{R}^d$ . As  $\hat{\mu} = 0$  on  $H_u$  and  $|\hat{\mu}|$  is invariant under  $RR'$ , it follows that  $\hat{\mu} = 0$  on a dense set. As  $\hat{\mu}$  is continuous,  $\hat{\mu} = 0$  on  $\mathbb{R}^d$  thus  $\mu = 0$ . Consequently,  $(S, H_1 \cup \dots \cup H_N)$  is a Heisenberg uniqueness pair.

(ii) Or  $G$  is finite. But then, there exists a subset  $W$  of  $\mathbb{R}^d$  (a Weyl chamber) such that  $\{gW : g \in G\}$  is a covering of  $\mathbb{R}^d$  and every  $x \in \mathbb{R}^d$  determines a unique  $g$  such that  $x \in gW$ . Now take any continuous function  $\varphi$  on  $\mathbb{S}^{d-1} \cap W$  and extend  $\varphi$  to  $\mathbb{S}^{d-1}$  by the following rule: if  $x \in \mathbb{S}^{d-1}$  then there exists a unique  $g \in G$  such that  $gx \in W$ . Writing  $g$  as  $g = \prod_{j=1}^M R_{k_j}$  with  $k_1, \dots, k_M \in \{1, \dots, N\}$ , we then set  $\varphi(x) = (-1)^M \varphi(gx)$ . The corresponding measure  $\mu(x) = \varphi(x) d\sigma(x)$  is well defined, non-zero, absolutely continuous with respect to surface measure, and satisfies  $R_k * \mu = -\mu$  by construction. By Lemma 2.2  $\hat{\mu} = 0$  on  $H_1 \cup \dots \cup H_N$ . Thus  $(S, H_1 \cup \dots \cup H_N)$  fails to be a Heisenberg uniqueness pair.  $\square$

**Remark 2.** When  $d = 2$  we recover the theorem of Lev and Sjölin: let  $\ell_1, \ell_2$  be two lines that intersect at 0. Then  $(\mathbb{S}^1, \ell_1 \cup \ell_2)$  is a Heisenberg uniqueness pair, if and only if the angle between  $\ell_1$  and  $\ell_2$  is not in  $\mathbb{Q}\pi$ .

However, when  $d \geq 3$ , there exists sets of 3 hyperplanes such that the angle between any two of them is in  $\mathbb{Q}\pi$  but such that the corresponding Coxeter group is infinite, see [RS] for a complete description of the 3-dimensional case.

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